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Introduction to Jet Schemes and Arc Spaces

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0 Introduction

1968 Nash, preprint (Nash problem)
1995 published in Duke Math [25]
1995 Kontsevich Lecture at Orsay [19]
Motivic integration $\Longrightarrow$ application
birational Calabi-Yau have the same Hodge number
Denef-Loeser [6][7] research on motivic integration
Mustaţă, Ein, Lazarsfeld, Yasuda
characterization of singularities by jet schemes [9][10][11][23][24], ([11] without motivic integration)

1 Jet Scheme and Arc Space

1.1 Existence

Notation. $k = \overline{k}$, char $k \geq 0$. X : variety$/k$, $\dim X = n$ (fixed).

Definition 1.1. Let $K \supset k$: field extension, $m \in \mathbb{Z}_{\geq 0}$.
A k-morphism $\alpha : \operatorname{Spec} K[t]/(t^{m+1}) \longrightarrow X$ is called an m-jet of X.
More precisely, "K-valued m-jet of X".
A k-morphism $\alpha : \operatorname{Spec} K[[t]] \longrightarrow X$ is called an arc of X.
$\operatorname{Spec} K[[t]] = \{0\,(= \text{closed point}), \eta\,(= \text{generic point})\}$.
More precisely, "K-valued arc of X".

Theorem 1.2. *X : variety$/k$. For $\forall m$, $\exists X_m$: k-scheme of finite type$/k$ s.t.*

$$\operatorname{Hom}_k(Z \times \operatorname{Spec} k[t]/(t^{m+1}), X) \cong \operatorname{Hom}_k(Z, X_m)$$

for $\forall Z$: k-scheme. Here $\operatorname{Hom}_k(Z \times \operatorname{Spec} k[t]/(t^{m+1}), X)$: space of m-jets parametrized by Z. "$\times$" means "$\times_{\operatorname{Spec} k}$".
In particular, in case $Z = \operatorname{Spec} K$ ($K \supset k$, field extension).

$$\begin{array}{ccc}
\operatorname{Hom}(\operatorname{Spec} K[t]/(t^{m+1}), X) & \cong & \operatorname{Hom}(\operatorname{Spec} K, X_m) \\
\Psi & & \Psi \\
\alpha & \longleftrightarrow & \alpha \ \text{(use the same notation)} \\
K\text{-valued } m\text{-jet of } X & & K\text{-valued point of } X
\end{array}$$

Proof. There are three proofs: (1) See BLR [4]. (2) Concrete construction. (3) See Vojta [33], more general construction, by "Hasse-Schmidt derivation". For X : S-scheme not necessary of finite type, $\forall S$: scheme.

(2) Here we show the second proof. $X = \operatorname{Spec} R$, $R = k[x_1, \dots, x_N]/(f_1, \dots, f_r)$. $Z = \operatorname{Spec} A$.

$$
\begin{aligned}
&\operatorname{Hom}_k(Z \times \operatorname{Spec} k[t]/(t^{m+1}), X) \\
&\quad = \operatorname{Hom}_k(k[x_1, \dots, x_N]/(f_1, \dots, f_r), A[t]/(t^{m+1})) \\
&\quad = \{\varphi : k[x_1, \dots, x_N] \longrightarrow A[t]/(t^{m+1}) \mid \varphi(f_i) = 0 \text{ for } \forall i\} \\
&\qquad \varphi : x_j \longmapsto a_j^{(0)} + a_j^{(1)} t + \cdots + a_j^{(m)} t^m, \quad a_j^{(\ell)} \in A \\
&\qquad \varphi(f_i) = F_i^{(0)}(a_j^{(\ell)}) + F_i^{(1)}(a_j^{(\ell)}) t + \cdots + F_i^{(m)}(a_j^{(\ell)}) t^m, \\
&\qquad\qquad F_i^{(m)}(a_j^{(\ell)}) \ : \text{ polynomial in } a_j^{(\ell)} \\
&\qquad \varphi(f_i) = 0 \Longrightarrow F_i^{(0)}(a_j^{(\ell)}), F_i^{(1)}(a_j^{(\ell)}), \cdots, F_i^{(m)}(a_j^{(\ell)}) = 0 \\
&\quad = \{\varphi : k[x_1, \dots, x_N, x_1^{(1)}, \dots, x_N^{(1)}, \dots, x_N^{(m)}] \longrightarrow A \mid F_i^{\ell'}(a_j^{(\ell)}) = 0\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (x_j^{(\ell)} \longmapsto a_j^{(\ell)}) \\
&\quad = \operatorname{Hom}(\underbrace{k[x_j, x_j^{(1)}, \dots, x_j^{(m)}]/F_i^{(\ell')}(x_j^{(\ell)})}_{=:R_m}, A) \\
&\quad = \operatorname{Hom}(\operatorname{Spec} A, \operatorname{Spec} R_m), \quad X_m = \operatorname{Spec} R_m
\end{aligned}
$$

□

How to get equations of $X_m \subset \mathbb{A}_k^M$ ($\operatorname{char} k = 0$). Derivation D on $k[x_j, x_j^{(1)}, \dots, x_j^{(m)}]$ is defined as follows:

$$D(x_j^{(\ell)}) = x_j^{(\ell+1)} \ (\ell < m), \quad D(x_j^{(m)}) = 0 \ (\text{otherwise}).$$

By the embedding

$$
\begin{array}{ccc}
X_m & \longrightarrow & \mathbb{A}^M \\
\Psi & & \Psi \\
\alpha & \longmapsto & (j!\, a_j^{(\ell)}),
\end{array}
$$

we have equations $\{D^j(f_i)\}$ of $X_m \subset \mathbb{A}^M$.

Note 1.3.

- X : affine $\Longrightarrow X_m$: affine
- X : finite type over $k \Longrightarrow X_m$: finite type over k.

Example 1.4.

- X : reduced variety, $\dim X = 0 \Longrightarrow X_m \cong X$.
- $X = \mathbb{A}^N \Longrightarrow X_m = \mathbb{A}_k^{(m+1)N}$.

Definition 1.5. We define $\psi_{m,m-1} : X_m \longrightarrow X_{m-1}$ as follows. Let $\alpha \in X_m$, $f \in A$. $\alpha(f) := a_0 + a_1 t + \cdots + a_{m-1}t^{m-1} + a_m t^m$. $\alpha' = \psi_{m,m-1}(\alpha)$ is defined by $\alpha'(f) := a_0 + \cdots + a_{m-1}t^{m-1}$.

More formally,

$$k[t]/(t^{m+1}) \longrightarrow k[t]/(t^m). \qquad \cdots\cdots\cdots (*)$$

(**) $\forall Z$: k-scheme, we have

$$\begin{array}{rcl} Z \times \operatorname{Spec} k[t]/(t^{m+1}) & \longleftarrow & Z \times \operatorname{Spec} k[t]/(t^m) \\ \operatorname{Hom}(Z \times \operatorname{Spec} k[t]/(t^{m+1}), X) & \longrightarrow & \operatorname{Hom}(Z \times \operatorname{Spec} k[t]/(t^m), X) \\ = \operatorname{Hom}(Z, X_m) & & = \operatorname{Hom}(Z, X_{m-1}). \end{array}$$

Put $Z := X_m$. We get

$$\begin{array}{ccc} \operatorname{Hom}(X_m, X_m) & \longrightarrow & \operatorname{Hom}(X_m, X_{m-1}) \\ \Downarrow & & \Downarrow \\ \mathrm{id} & \longmapsto & \psi_{m,m+1}. \end{array}$$

We say this argument (**) is the argument "induced from (*)".

Let $m' > m$. Define $\psi_{m',m} := \psi_{m+1,m} \circ \cdots \circ \psi_{m',m'-1}$.

Example 1.6. X : non-singular variety, $\psi : X_{m'} \longrightarrow X_m$ locally trivial fibration with the fiber $\mathbb{A}^{(m'-m)n}$.

In case $X = \mathbb{A}^n_k$, we have

$$\begin{array}{ccc} X_{m'} = \mathbb{A}_k^{(m'+1)n} & \longrightarrow & X_m = \mathbb{A}_k^{(m+1)n}, \\ \Downarrow & & \\ \alpha & & \\ \alpha(x_i) = \sum_{j=0}^{m'} a_{ij} t^j & \longmapsto & \sum_{j=0}^{m} a_{ij} x^j \\ k[x_j, x_j^{(1)}, \ldots, x_j^{(m')}] & \hookleftarrow & k[x_j, x_j^{(1)}, \ldots, x_j^{(m)}] \\ \mathbb{A}^{(m'+1)n} & \longrightarrow & \mathbb{A}^{(m+1)n} \quad \text{canonical projection.} \end{array}$$

Definition 1.7. We define

$$\begin{array}{cccc} \pi_m : & X_m & \longrightarrow & X \\ & \Downarrow & & \Downarrow \\ & \alpha & \longmapsto & \alpha(0) \end{array} \qquad \left(\begin{array}{ccc} \operatorname{Spec} K[t]/(t^{m+1}) & \longrightarrow & X \\ \Downarrow & & \Downarrow \\ \{0\} & \longmapsto & \alpha(0) \end{array} \right),$$

induced from (*), $k[t]/(t^{m+1}) \longrightarrow k$ (discussion as before).

Example 1.8. Even if X is irreducible, X_m is not necessarily irreducible.

For example, $X = \{x^2 - y^2 + x^3 = 0\} \subset \mathbb{C}^3 \Longrightarrow X_1 = Z_1 \cup Z_2$ irreducible decomposition. Here $\pi_1^{-1}(0) = \mathbb{A}^2 = Z_1$, $\pi_1^{-1}(X_{\mathrm{reg}}) = \mathbb{A}^1$-bundle. Z_2 : closure of $\pi_1^{-1}(X_{\mathrm{reg}})$.

Definition 1.9. $m' > m \Rightarrow \psi_{m',m} : X_{m'} \longrightarrow X_m$ projective system. From

$$X_{m'} \xrightarrow{\psi_{m',m}} X_m \xrightarrow{\psi_{m,m''}} X_{m''},$$

we can define $X_\infty := \varprojlim_{m\to\infty} X_m$. Note that $X = \operatorname{Spec} R \Longrightarrow X_m = \operatorname{Spec} R_m$. Put $\varinjlim R_m =: R_\infty$, $X_\infty := \operatorname{Spec} R_\infty$.

Theorem 1.10. *For any k-scheme Z,*

$$\operatorname{Hom}_k(Z \hat{\times} \operatorname{Spec} k[[t]], X) \cong \operatorname{Hom}_k(Z, X_\infty).$$

$\because$) $\operatorname{Hom}_k(Z \times \operatorname{Spec} k[[t]]/(t^{m+1}), X) \cong \operatorname{Hom}_k(Z, X_m)$. Taking projective limit $m \to \infty$, we have $\operatorname{Hom}_k(Z \hat{\times} \operatorname{Spec} k[[t]], X) = \operatorname{Hom}_k(Z, X_\infty)$.

Put $Z = \operatorname{Spec} A$. We obtain $\operatorname{Hom}_k(\operatorname{Spec} A[[t]], X) \cong \operatorname{Hom}_k(\operatorname{Spec} A, X_\infty)$. NB. $A[[t]] \neq A \otimes k[[t]]$. For example, $A = k[x]$.

Example 1.11. $X = \mathbb{A}_k^n$. $X_\infty = \operatorname{Spec} k[x_j, \ldots, x_j^{(1)}, \ldots, x_j^{(2)}, \ldots] =: \mathbb{A}_k^\infty$.

$$\begin{aligned} m \in \mathbb{N} &: \{\text{closed point of } \mathbb{A}_k^m\} = k^m, \\ m = \infty &: \{\text{closed point of } \mathbb{A}_k^\infty\} \neq k^\infty (\text{if } \sharp k = \aleph_0). \end{aligned}$$

Definition 1.12. Define a k-morphism as follows:

$$\begin{array}{cccc} \psi_m : X_\infty & \longrightarrow & X_m \\ \Downarrow & & \Downarrow \\ \alpha & \longmapsto & \alpha_m \end{array}$$

$$\alpha(f) = \sum_{j=0}^{\infty} a_j t^j, \qquad \alpha_m(f) = \sum_{j=0}^{m} a_j t^j.$$

Formally this morphism is "induced from" $k[[t]] \longrightarrow k[[t]]/(t^{m+1})$. Here α is

$$\text{arc } \alpha : \operatorname{Spec} K[[t]] \longrightarrow \operatorname{Spec} A \subset X \quad \longleftrightarrow \quad \text{ring hom } K[[t]] \xleftarrow{\alpha} A.$$

Define a k-morphism

$$\begin{array}{cccc} \pi : & X_\infty & \longrightarrow & X \\ & \Downarrow & & \Downarrow \\ & \alpha & \longmapsto & \alpha(0) \end{array}$$

where 0 is the closed point of $\operatorname{Spec} K[[t]]$. Formally this morphism π is "induced from" $k[[t]] \to k$.

We have $X_\infty \xrightarrow{\psi_{m'}} X_{m'} \xrightarrow{\psi_{m',m}} X_m \xrightarrow{\pi_m} X$, $X_\infty \xrightarrow{\pi} X$. If X : smooth then $\psi_{m'}$: surjective.

Proposition 1.13. *$\psi_m(X_\infty)$ is constructible set (finite union of locally closed subsets).*

1.2 Functoriality

Proposition 1.14. *Let $m \in \mathbb{N} \cup \{\infty\}$, $f : X \longrightarrow Y$ k-morphism.*

$$\begin{array}{ccc} \operatorname{Spec} K[t]/(t^{m+1}) & \xrightarrow{\alpha} & X \\ & \searrow & \downarrow f \\ & & Y \end{array}$$

Then

$$\begin{array}{rccccl} f_m : & X_m & \longrightarrow & Y_m & & k\text{-morphism} \\ & \Psi & & \Psi & & \\ & \alpha & \longmapsto & f \circ \alpha & & \\ & \pi_m^X \downarrow & & \downarrow \pi_m^Y & & \\ & X & \xrightarrow{f} & Y & & \end{array}$$

is commutative.

Proposition 1.15. *Let $m \in \mathbb{N} \cup \{\infty\}$. Then $X \xrightarrow{f} Y$ is étale $\Longrightarrow X_m \cong Y_m \times_Y X$.*

Corollary 1.16. *Let m be as above. Then $U \hookrightarrow X$: open immersion $\Longrightarrow U_m \cong (\pi_m^X)^{-1}(U)$. In particular, $U_m \hookrightarrow X_m$ is an open immersion.*

Definition 1.17. Main part of $X_m \overset{\text{def}}{\Longleftrightarrow} \overline{\pi_m^{-1}(X_{\text{reg}})}$

Note 1.18. $Z \subset X$ closed immersion $\Longrightarrow Z_m \subset X_m$ closed immersion.
$\because$) $Z \subset X \subset \mathbb{A}^N$. $\mathcal{O}_{\mathbb{A}^N} \supset \mathcal{I}_Z \supset \mathcal{I}_X$, $\mathcal{O}_{\mathbb{A}^{N(m+1)}} \supset \mathcal{I}_{Z_m} \supset \mathcal{I}_{X_m}$.

NB. $Z \subset X$ closed, $Z_m \subsetneq \pi_m^{-1}(Z)$, $Z_\infty \subsetneq \pi_\infty^{-1}(Z)$. (codim $Z_\infty = \infty$.)

Proposition 1.19. *Let $m \in \mathbb{N} \cup \{\infty\}$. Then $(X \times_{\operatorname{Spec} k} Y)_m = X_m \times_{\operatorname{Spec} k} Y_m$.*

Theorem 1.20 (Kolchin [18], Ishii-Kollár [14, Lemma 2.12]). *Let* char $k = 0$. *Then X : irreducible $\Longrightarrow X_\infty$: irreducible.*

Example 1.21. char $k = p > 0$. X_∞ is not necessarily irreducible. $X = \{x^p - y^p z = 0\} \subset \mathbb{A}_k^3 \Longrightarrow X_\infty$ is not irreducible. (See IK [14].)

Note 1.22.

- $f : X \longrightarrow Y$ surjective $\not\Rightarrow f_m : X_m \longrightarrow Y_m$ surjective,
- $f : X \longrightarrow Y$ proper $\not\Rightarrow f_m : X_m \longrightarrow Y_m$ proper.

Assume that X has A_n singularity $\subset \mathbb{A}^3$. Then f is proper surjective:

$$\begin{array}{ccccc} \mathbb{C}^2 & \xrightarrow{f} & \mathbb{C}^2/G = X & & \\ \uparrow & & \uparrow & & \\ \mathbb{C}_m^2 & \xrightarrow{f_m} & X_m & & f_m : \text{dominant}. \end{array}$$

X_m is irreducible (Mustaţă) and $\mathbb{C}^2 \ni 0 \overset{f}{\longmapsto} P \in X$, $(\pi_m^{\mathbb{C}^2})^{-1}$ is irreducible. But if $m \gg 0$, $(\pi_m^X)^{-1}(P)$ has n irreducible components (Nash). Therefore f_m is not surjective, in particular it is not proper.

1.3 Cylinders and valuation

Definition 1.23 (ELM[11]). Let $C \subset X_\infty$ be an irreducible constructible set.

$$C : \text{thin} \overset{\text{def}}{\Longleftrightarrow} \exists Z \subset X \text{ proper closed } (C \subset Z_\infty)$$
$$C : \text{fat} \overset{\text{def}}{\Longleftrightarrow} \text{not thin.}$$

Note 1.24. Let $\alpha \in C$ irreducible constructible set $\subset X$. Then

$$C : \text{fat} \iff \alpha(\eta) : \text{generic point of X}$$

$$\iff \begin{array}{ccc} k[[t]] & \overset{\alpha}{\hookleftarrow} & A \quad (X = \operatorname{Spec} A) \\ \cap & & \downarrow \\ K((t)) & \overset{\alpha}{\longleftarrow} & K(X) \quad \text{extendable,} \end{array}$$

where η is the generic point of $\operatorname{Spec} K[[t]]$.

Let $v_C(f) := \operatorname{ord} \alpha(f)$ $(f \in K(X)\backslash\{0\})$. Then v_C : discrete valuation.

Definition 1.25. $C \subset X_\infty$ is a cylinder $\overset{\text{def}}{\Longleftrightarrow} \exists S \subset X_\infty$ constructible set, $C = \psi_m^{-1}(S)$.

Note 1.26. X: non-singular, $C = \psi_m^{-1}(S)$: cylinder.

(0) $X_\infty \longrightarrow X_m$ locally trivial fibration with the fiber $\mathbb{A}^\infty$.

(1) $S = S_1 \cup S_2 \cup \cdots \cup S_r$ irreducible decomposition
$\Longrightarrow C = \psi_m^{-1}(S_1) \cup \cdots \cup \psi_m^{-1}(S_r)$ irreducible decomposition.

(1)' In particular, cylinder's irreducible components are finite.

(2) C : cylinder $\Longrightarrow \overline{C} = \psi_m^{-1}(\overline{S})$ cylinder.

(3) $\forall$ irreducible components of cylinder is fat.
Corresponding valuation is divisorial valuation, i.e., $\exists E$: divisor over X, $v_C = q \operatorname{val}_E$ $(q \in \mathbb{N})$.

If X has singularity, (1)' also holds but (0), (1) and (3) are not affirmative. (2) is open problem.

Example 1.27 (Thin cylinder [De Fernex-Ein-Ishii, preprint]). Let $F = x^2 - y^2 z$, $X = \{F = 0\} \subset \mathbb{A}^3_{\mathbb{C}}$, $\alpha_m \in X_m$ closed point.

$$\mathbb{C}[[t]]/(t^{m+1}) \xleftarrow{\alpha_m} \mathbb{C}[x,y,z]/(F), \quad \begin{cases} \alpha_m(x) & = t \\ \alpha_m(y) & = 0 \\ \alpha_m(z) & = 0 \end{cases}.$$

Then cylinder $\psi_m^{-1}(\alpha_m) \subset (\operatorname{Sing} X)_\infty$ is thin!

Proposition 1.28 (De Fernex-Ein-Ishii). X : *singular.*

(1) $\sharp$(components of cylinder) $< \infty$.

(2) A thin component of a cylinder $\subset (\operatorname{Sing} X)_\infty$.

(3) C : fat component of a cylinder $\Longrightarrow v_C$: divisorial valuation.

Proposition 1.29. C : *cylinder* $\Longrightarrow \forall m \in \mathbb{N}$, $\psi_m(C)$ *is a constructible set.*

2 Motivic Integration

Exposition text Craw [5], Veys [32], Loeser [8].
On Nash problem, see [3],[12],[14]–[17],[20]–[22],[27],[28],[30],[31].

2.1 Grothendieck ring

$\mathcal{V}ar_{\mathbb{C}} :=$ {variety $/\mathbb{C}$}. $K_0(\mathcal{V}ar_{\mathbb{C}}) :=$ abelian group generated by $\{[V] \mid V \in \mathcal{V}ar_{\mathbb{C}}\}/$(equiv.), equiv. means as follows:
$[V] = [W]$ if $V \cong W$, $[V] = [V \setminus Z] + [Z]$, $Z \subset V$ closed.
This has a multiplication $[V][W] := [V \times W]$.

$K_0(\mathcal{V}ar_{\mathbb{C}}) =$ "Grothendieck ring."
$\forall C$: constructible set in some variety V. $C = \coprod A_i \Longrightarrow$ naturally $[C] := \sum [A_i] \in K_0(\mathcal{V}ar_{\mathbb{C}})$. Here A_i : locally closed.
Convention: [point]=: 1, $[\mathbb{A}^1] =: \mathbb{L}$.

Example 2.1.

(1) $X = \{y^2 - x^3 = 0\} \in \mathbb{A}^2 \Longrightarrow [X] = [\mathbb{A}^1 \setminus \{0\}] + [\{0\}] = [\mathbb{A}^1] = \mathbb{L}$.

(2) $\forall f : Y \longrightarrow X$ piecewise trivial fibration with fiber F, i.e., $X = \coprod X_i$ locally closed and

$$f|_{f^{-1}(X_i)} \;:\; \underset{\cong X_i \times F}{f^{-1}(X_i)} \longrightarrow X_i.$$

then $[Y] = [X][F]$.
$\because$) $[Y] = \sum [f^{-1}(X_i)] = [F] \sum [X_i] = [F][X]$, since $[f^{-1}(X_i)] = [X_i] \times [F]$, $\sum [X_i] = [X]$.

Note 2.2. $K_0(\mathcal{V}ar_{\mathbb{C}})$ is not integral domain. See Poonen [29]. $\exists A, B$: Abelian varieties, $[A] \neq [B]$ and $A \times A \cong B \times B \Longrightarrow ([A]-[B])([A]+[B]) = [A]^2 - [B]^2 = 0$.

Definition 2.3 (Hodge-Deligne polynomial). Let $V \in \mathcal{V}ar_{\mathbb{C}}$, $\dim V = n$.

$$H(V,u,v) := \sum_{p,q=0}^{n} \sum_{i=0}^{2n} (-1)^i h^{pq}\big(H^i_C(V,\mathbb{C})\big) u^p v^q \in \mathbb{Z}[u,v]$$

where h^{pq} is dimension of (p,q)-Hodge components. In particular, $H(V,1,1) = \chi(V)$ Euler characteristic.

Note 2.4.

$$\begin{array}{ccc} \mathcal{V}ar_{\mathbb{C}} & \xrightarrow{H} & \mathbb{Z}[u,v] \quad \text{factors.} \\ \searrow & & \nearrow H \\ & K_0(\mathcal{V}ar_{\mathbb{C}}) & \end{array}$$

Example 2.5.

(1) $H(\mathbb{P}^n,u,v) = 1 + uv + (uv)^2 + \cdots + (uv)^n$.

$\because)$ $h^{pq}\big(H^i(\mathbb{P}^n,\mathbb{C})\big) = \begin{cases} 1 & (p = q = \frac{i}{2}) \\ 0 & (\text{otherwise}) \end{cases}$.

(2) $H(\mathbb{A}^n,u,v) = (uv)^n$. In particular, $H(\mathbb{L},u,v) = uv$.

$$\begin{array}{lccc} \because) & [\mathbb{P}^n] & = & [\mathbb{A}^n] + [\mathbb{P}^{n-1}] \\ & 1 + uv + \cdots + (uv)^n & & (uv)^n \qquad 1 + uv + \cdots + (uv)^{n-1} \end{array}$$

Definition 2.6. $\mathcal{M} := K_0(\mathcal{V}ar_{\mathbb{C}})_{\mathbb{L}}$, localization by $\mathbb{L}$.

Definition 2.7. $F^m :=$ subgroup generated by $\frac{[S]}{\mathbb{L}^i}$. $\dim S - i \le -m$. $\Longrightarrow \{F^m\}$ descending filtration, $F^m \cdot F^n \subseteq F^{m+n}$. $\mathcal{M}_{\mathbb{C}}/F^m \longrightarrow \mathcal{M}_{\mathbb{C}}/F^{m-1}$: projective system. $\widehat{\mathcal{M}}_{\mathbb{C}} := \varprojlim_m \mathcal{M}_{\mathbb{C}}/F^m$,

We say "$\sum_{m\in\mathbb{Z}} a_m \mathbb{L}^{-m}$ converge" ($a_m \in \mathcal{V}ar_{\mathbb{C}}$) $\Longleftrightarrow \sum_{m\in\mathbb{Z}} a_m\mathbb{L}^{-m} \in \widehat{\mathcal{M}}_{\mathbb{C}}$.

$$\Big(\Longleftrightarrow \sum_{\dim a_m - m > k} a_m \mathbb{L}^{-m} \in \widehat{\mathcal{M}}_{\mathbb{C}}/F^k.\Big)$$

For example, $\sum_{m\in\mathbb{Z}} \mathbb{L}^{-m}$ does not converge.

2.2 Motivic integration

There are two ways.
(1) in $\widehat{\mathcal{M}}_C$, Denef, Loeser, Veys. (2) in $\mathbb{Z}[[u^{-1}v^{-1}]][u,v]$, Mustaţǎ.

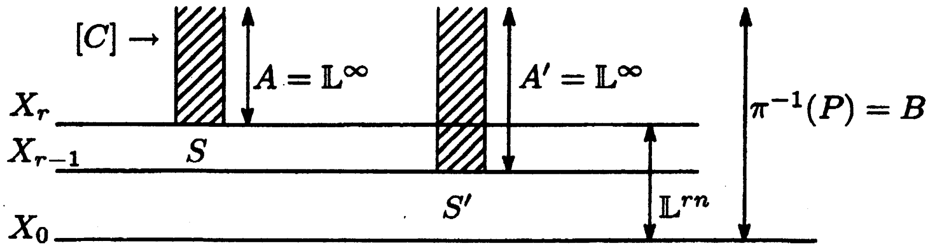
(1) C : cylinder $\subset X_\infty$, $n = \dim X$.

$$\text{motivic measure } \mu(C) = \lim_{n\to\infty} \frac{[\psi_m(C)]}{\mathbb{L}^{(m+1)n}} \in \widehat{\mathcal{M}}_{\mathbb{C}}. \quad \text{(DL[6])}$$

$\Big($Note that Veys defines $\mu(C) = \lim\limits_{n\to\infty} \dfrac{[\psi_m(C)]}{\mathbb{L}^{mn}}$.$\Big)$
In case X is non-singular, $C = \psi_r^{-1}(S)$, $S \subset X_r$.
$m > r$, $\psi_m(C) = \psi_{m,r}^{-1}(S)$, $[\psi_m(C)] = [S]\mathbb{L}^{(m-r)n}$.

$$\frac{[\psi_m(C)]}{\mathbb{L}^{(m+1)n}} = \frac{[S]}{\mathbb{L}^{(r+1)n}} \quad \text{for } \forall m. \quad \mu(C) = \frac{[S]}{\mathbb{L}^{(r+1)n}}.$$

Picture



$$\text{“}[C]\text{”} = [S]A = [S]B/\mathbb{L}^{rn} \Rightarrow \frac{[S]}{\mathbb{L}^{rn}}$$

Here by putting $B = 1$, we can think that $\frac{[S]}{\mathbb{L}^{rn}}$ is a “volume” of C.

Example 2.8 (Singular case). $X = \{xy = 0\} \subset \mathbb{A}^2$. X_∞ : cylinder, $[\psi_m(X_\infty)] = 2\mathbb{L}^{m+1} - 1$. When $m = 1$, $\psi_1(X_\infty) =$ horizontal part. $\mathbb{A}^2 \cup \mathbb{A}^2$, $\mathbb{L}^2 + \mathbb{L}^2 - 1 = 2\mathbb{L}^2 - 1$.

$$\mu(X_\infty) = \lim_{m\to\infty} \frac{[\psi_m(X_\infty)]}{\mathbb{L}^{m+1}} = \lim_{m\to\infty} \frac{2\mathbb{L}^{m+1} - 1}{2\mathbb{L}^{m+1}} = 2.$$

Definition 2.9 (Motivic Integration). $F : X_\infty \longrightarrow \mathbb{Z} \cup \{\infty\}$ function s.t. $F^{-n}(n)$ is a cylinder.

$$\int_{X_\infty} \mathbb{L}^{-F} d\mu := \sum_{m\in\mathbb{Z}} \mu(F^{-1}(n))\mathbb{L}^{-n} \in \widehat{\mathcal{M}_{\mathbb{C}}}$$

Example 2.10 (Example of F). $Z \subset X = \operatorname{Spec} A$ closed subscheme. F_Z : $X_\infty \longrightarrow \mathbb{Z} \cup \{\infty\}$ $\big(\alpha \longmapsto \operatorname{ord}_\alpha(Z) = \operatorname{ord}_t \alpha(I_Z)\big)$ satisfies the condition for F.
$\because)$ $F_Z(\mathbb{Z}_{\geq m}) = \psi_{m-1}^{-1}(Z_{m-1})$ (Exercise). $F_Z^{-1}(m) = \psi_{m-1}^{-1}(Z_{m-1}) \setminus \psi_m^{-1}(Z_\infty)$.

(2) by Mustaţă. F : as above.

$$\begin{aligned}\int_{X_\infty} e^{-F} &:= \sum_{m\in\mathbb{Z}} H(\mu(F^{-1}(n))\mathbb{L}^{-n}) \in \mathbb{Z}[[u^{-1}v^{-1}]][u,v] \\ &= \sum_{m\in\mathbb{Z}} H\big(\mu(F^{-1}(n))\big)(uv)^{-n} \in \mathbb{Z}[[u^{-1}v^{-1}]][u,v].\end{aligned}$$

Theorem 2.11 (Change of variables formula, DL [6]). $\varphi : Y \longrightarrow X$ *proper birational morphism of non-singular varieties. Then*

$$\int_{X_\infty} \mathbb{L}^{-F} d\mu = \int_{Y_\infty} \mathbb{L}^{-F\circ\varphi_\infty - F_{K_{Y/X}}} d\mu.$$
$$\int_{X_\infty} e^{-F} = \int_{Y_\infty} e^{-F\circ\varphi_\infty - F_{K_{Y/X}}}.$$

Corollary 2.12. X, X' : *smooth Calabi-Yau varieties.* $X \sim X'$ *birational* $\Longrightarrow [X] = [X']$.

Proof. $K_{Y/X} = K_{Y/X'} = K_Y$.

$$\begin{array}{ccc} & Y & \\ {}^{\varphi}\swarrow & & \searrow^{\varphi'} \\ X & & X' \end{array}$$

$X_\infty \xrightarrow{F} \mathbb{Z} \cup \{\infty\}$ (zero map), $X'_\infty \xrightarrow{F'} \mathbb{Z} \cup \{\infty\}$ (zero map) $\Longrightarrow F \circ \varphi_\infty = F' \circ \varphi_\infty$: zero map.

$$\int_{X_\infty} \mathbb{L}^{-F} d\mu = \mu(F^{-1}(0)) = \frac{[\pi(X_\infty)]}{\mathbb{L}^{(0+1)n}} = \frac{[X]}{\mathbb{L}^n},$$

$$\int_{Y_\infty} \mathbb{L}^{-F\circ\varphi_\infty - F_{K_{Y/X}}} = \int_{X'_\infty} \mathbb{L}^{-F'} d\mu = \frac{[X']}{\mathbb{L}^n} \Longrightarrow [X] = [X'] \in \widehat{\mathcal{M}}_{\mathbb{C}}.$$

□

Corollary 2.13. X, X' : *birational Calabi-Yau varieties* $\Longrightarrow h^{pq}(X) = h^{pq}(X')$.

2.3 Application of motivic integration

Characterization of singularities via jets.

Theorem 2.14 (Mustaţă, [23]). X : *smooth,* $Y \subset X$ *closed subscheme. Then*

$$\text{log canonical threshold } c(X,Y) = \dim X - \sup_{m\geq 1} \frac{\dim Y_m}{m+1}.$$

Theorem 2.15 (Ein-Mustaţă-Yoshida, [9]). X : *locally complete intersection variety* (lci *for short*). *Then*

X has canonical singularity $\Longleftrightarrow$ *X_m is irreducible for* $\forall m \in \mathbb{N}$.

($\Longleftrightarrow X$: *rational in this situation*)

Theorem 2.16 (Ein-Mustaţă, [10]). X : *normal lci.* $Y = \sum a_i Y_i$ $(a_i \in \mathbb{R})$, $Y_i \subset X$ *irreducible closed subscheme. Then* $x \longmapsto \mathrm{mld}\,(x; X, Y)$ (minimal log discrepancy) *is upper semi-continuous.*

We show Mustaţă's proof of Theorem 2.14.

Theorem 2.14'(M[23]). X : smooth, $Y \subset X$ closed. Then
(X, qY) is log canonical $\iff \dim Y = (m+1)(\dim X - q)$, $\forall m \in \mathbb{N}$.

Theorem 2.14' $\Longrightarrow$ Theorem 2.14.
$\because$) log canonical threshold $= \sup\{q \mid (X, qY) : \text{log canonical}\}$. $\square$

Let $\varphi : X' \longrightarrow X$ be a log resolution of (X, Y). $\varphi^{-1}(Y) = \sum_{i=1}^{r} a_i D_i$, $a_i \geq 1$, $D_i \in X'$ irreducible divisor. $K_{X'/X} = \sum_{i=1}^{r} b_i D_i$, $b_i \geq 0$. Then

$$\begin{aligned}(X, qY) : \text{ log canonical} &\iff b_i - qa_i \geq -1 \quad (i = 1, \ldots, r)\\ &\iff qa_i - b_i - 1 \leq 0.\end{aligned}$$

Here (X, qY) : log canonical $\overset{\text{def}}{\iff}$

$$K_{X'} = \varphi^* K_X + q\varphi^{-1}(Y) + \sum r_i D_i, \quad r_i \geq -1 \quad \text{for } \forall i.$$

From this,

$$\begin{array}{ccc} K_{X'} - \varphi^* K_X & = & q\varphi^{-1}(Y) + \sum r_i D_i \\ \| & & \| \\ K_{X'/X} = \sum b_i D_i & & q \sum a_i D_i, \qquad b_i = qa_i + r_i,\ r_i \geq -1. \end{array}$$

Recall motivic integration.

$$X_\infty \xrightarrow{F_Y} \mathbb{Z} \cup \{\infty\} \xrightarrow{f} \mathbb{Z} \cup \{\infty\}.$$

where $F^{-1}(s)$: cylinder for $\forall s \in \mathbb{Z}$. We define f later.
We have two expressions

$$\int_{X_\infty} e^{-F} = \int_{X'_\infty} e^{-F \circ \varphi_\infty - F_{K_{X'/X}}}.$$

We compare the left and right of this equality.

NB. $F_Y^{-1}(m) = \psi_{m-1}^{-1}(Y_{m-1}) \setminus \psi_m^{-1}(Y_m)$.

$$\mu(F_Y^{-1}(m)) = \frac{[Y_{m-1}]}{\mathbb{L}^{mn}} - \frac{[Y_n]}{\mathbb{L}^{(m+1)n}}.$$

Put $f(m) = s$.

$$\begin{aligned}\int_{X_\infty} e^{-F} &= \sum_{s \in \mathbb{Z}_{\geq 0}} H(\mu(F^{-1}(s)))(uv)^{-s} = \sum_{m \in \mathbb{Z}_{\geq 0}} H(\mu(F_Y^{-1}(m)))(uv)^{-f(m)}\\ &\qquad \uparrow F^{-1}(s) = F_Y(f^{-1}(s))\\ &= \sum_{m \in \mathbb{Z}_{\geq 0}} \left(H(Y_{m-1})(uv)^{-mn} - H(Y_m)(uv)^{-(m+1)n}\right)(uv)^{-f(m)}\\ &= \sum H(Y_{m-1})(uv)^{-mn-f(m)} - \sum H(Y_m)(uv)^{-(m+1)n-f(m)}\\ &=: S_1 - S_2.\end{aligned}$$

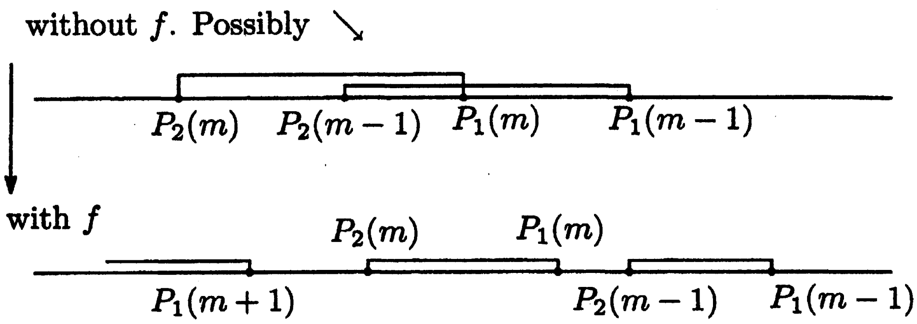
Definition 2.17 (Definition of f). We define $f : \mathbb{Z} \cup \{\infty\} \hookrightarrow \mathbb{Z} \cup \{\infty\}$ as follows:

$$\begin{cases} m \leq 0: & f(m) = m \\ m > 0: & \text{inductively, } f(m+1) > f(m) + \dim Y_m + c(m+1), \end{cases}$$

where $c > \left| n - \dfrac{b_i - 1}{a_i} \right|$ for $\forall i$.

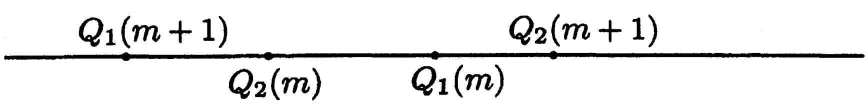
Now, looking at S_1, we have

$$\underset{=: 2P_2(m)}{2(-mn - f(m))} < \left\{ \begin{matrix} \text{deg of monomials in} \\ H(Y_{m-1})(uv)^{-mn-f(m)} \end{matrix} \right\} < \underset{=: 2P_1(m)}{2(-mn - f(m) + \dim Y_{m-1})}.$$



Next, looking at S_2, we have

$$\underset{=: 2Q_2(m)}{2(-(m+1)n - f(m))} < \left\{ \begin{matrix} \text{deg of monomials in} \\ H(Y_m)(uv)^{-(m+1)n-f(m)} \end{matrix} \right\} < \underset{=: 2Q_1(m)}{2(-(m+1)n - f(m) + \dim Y_m)}.$$



Compare $Q_1(m)$, $P_1(m)$. We have

$$P_1(m) \geq Q_1(m).$$

Here, "=" holds $\Longleftrightarrow$ $\dim Y_m = \dim Y_{m-1} + n$.
$\because$) By $\dim Y_m \leq \dim Y_{m-1} + n$ the inequality of P_1 and Q_1 follows. $\square$

Put $\ell(Y_{m-1}) = \sharp$(maximal dim components). Then
- ∘ The term of deg $= -2P_1(m)$ in $S_1 = \ell(Y_{m-1})(uv)^{-P_1(m)}$ (Exercise).
- ∘ The term of deg $= -2Q_1(m)$ in $S_2 = \ell(Y_m)(uv)^{-Q_1(m)}$.

Change of variables formula,

$$\int_{X'_\infty} e^{-F\circ\varphi_\infty - F_{K_{X'/X}}} = \sum_{J\subset\{1,\dots,r\}} S_J, \quad \text{(For "$=$", see Batyrev[1], Theorem 36)}$$

$$S_J = \sum H(D_j^0)(uv-1)^{|J|}(uv)^{-n-\sum \alpha_i(b_i+1) - f(\sum \alpha_i a_i)} =: \sum(\star),$$

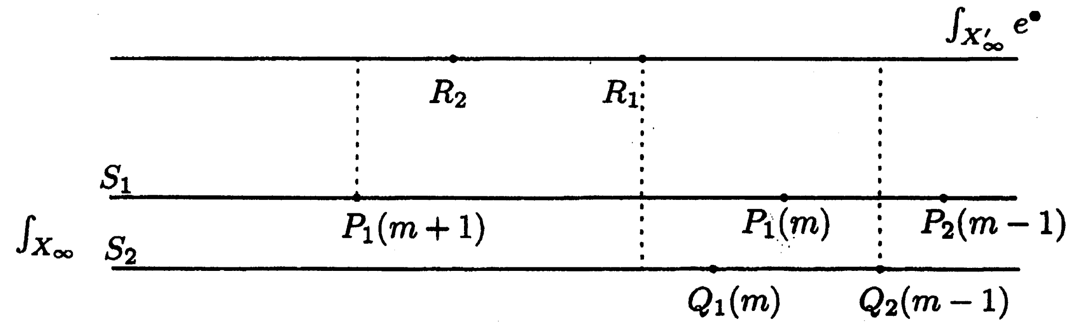
$$D_J = \Big(\bigcap_{i\in J} D_i\Big) \setminus \bigcup_{i\notin J} D_i.$$

We have

$$\underbrace{2R_1(\alpha_i \,|\, i\in J) - 2n}_{=:\,2R_2(\alpha_i|\,i\in J)} < \text{ deg of } (\star) \ < \underbrace{2\sum_{i\in J}(\alpha_i(b_i+1) - f(\sum \alpha_i a_i))}_{=:\,2R_1(\alpha_i|\,i\in J)}$$

Put $\tau(m) = \dim Y_m - (m+1)(n-q)$.

$$R_1(\alpha_i|\, i\in J) =$$
$$P_1(\textstyle\sum\alpha_i a_i) - \tau\underbrace{(\textstyle\sum\alpha_i a_i - 1)}_{=:(*)} + \underbrace{\textstyle\sum\alpha_i(q a_i - b_i - 1)}_{=:(**)} \cdots\cdots (1)$$
$$P_1(\textstyle\sum\alpha_i a_i + 1) < R_2(\alpha_i|\, i\in J)$$
$$< R_1(\alpha_i|\, i\in J) < \min\{Q_2(\textstyle\sum\alpha_i a_i - 1), P_2(\textstyle\sum\alpha_i a_i - 1)\}$$



Assume (X, qY) : log canonical. Then $(**) \le 0$.

If $\exists m$ s.t. $\tau(m-1) > 0 \iff (*) > 0$ for $m = \sum \alpha_i a_i$
$\Longrightarrow R_1(\alpha_i \mid i\in J) < P_1(\sum\alpha_i a_i)$.
$\Longrightarrow (uv)^{P_1(m)}$ does not appear in $\int_{X_\infty} e^{-F}$. Therefore, the monomial $(uv)^{P_1(m)}$ in S_1 should be cancelled by a term of S_2.
$\Longrightarrow P_1(m) = Q_1(m) \Longrightarrow \dim Y_m = \dim Y_{m-1} + n$.

$\tau(m) = \tau(m+1) - q > 0 \overset{\text{same argument}}{\Longrightarrow} P_1(m+1) = Q_1(m+1)$

$$\Longrightarrow \dim Y_{m+1} = \dim Y_m + n \Longrightarrow \tau(m+1) = \tau(m+2) - q > 0 \Longrightarrow \cdots\cdots$$

Therefore, $\exists m_0$ s.t. $\forall m \geq m_0 \Longrightarrow \dim Y_m = \dim Y_{m-1} + n$
$\Longrightarrow Y_\infty \supset \psi_m^{-1}(Y_m^0)$, Y_m^0 : maximal dim component.

Y_∞ : thin, $\psi_m^{-1}(Y_m^0)$: cylinder, fat. "thin $\supset$ fat" : contradiction.

(The proof of the converse was not shown in the talk because of the shortness of time. One who are interested in can see it in Mustaţă's paper [23]. Here we see just the sketch of it.)

Assume $\tau(m) \leq 0$ for all $m \in \mathbb{N}$. Fix m such that $a_i \mid m+1$ for every i. Here, if $qa_j - b_j - 1 > 0$ for some j, define

$$J = \{j\} \text{ and } \alpha_i = \begin{cases} \frac{m+1}{a_j} & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

Then,

$$\begin{aligned} &\Longrightarrow \quad R_1(\alpha_i \mid i \in J) = P_1(m+1) - \underbrace{\tau(m)}_{\leq 0} + \underbrace{\tfrac{m+1}{a_j}(qa_j - b_j - 1)}_{>0} \\ &\Longrightarrow \quad P_1(m+1) < R_1(\alpha_i \mid i \in J). \end{aligned}$$

By this and some other discussions, it follows that there is d in the interval $(P_1(m+1), \min\{P_2(m), Q_2(m)\})$ such that the term $(uv)^{-d}$ appears in $\int_{X'_\infty} e^{-F \circ \varphi_\infty - F_{X'/X}}$. But for any d in this interval, the term $(uv)^{-d}$ does not appear in $\int_{X_\infty} e^{-F}$, a contradiction. Therefore the inequality $qa_i - b_i - 1 \geq 0$ should hold for every i, i.e., (X, qY) is log-canonical. $\square$

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